This is a version of an old exam, with a couple questions replaced. Note: many topics are still not represented!

1. (a) Solve the following system of linear equations, putting your answer in parametric vector form:

\[
\begin{align*}
2x_1 + 2x_2 + 2x_3 + x_4 &= 12 \\
-2x_1 - 2x_2 + x_3 &= -1 \\
x_1 + x_2 + 2x_3 + x_4 &= 10
\end{align*}
\]

(b) Consider the matrix

\[
A = \begin{bmatrix}
11 & 6 & 17 & 28 \\
-1 & -1 & -2 & -3 \\
3 & 2 & 5 & 8
\end{bmatrix}
\]

with reduced echelon form

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Given that

\[
A \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix} = \begin{bmatrix}
186 \\
-21 \\
54
\end{bmatrix},
\]

find the parametric vector form the solutions to \(Ax = \begin{bmatrix}
186 \\
-21 \\
54
\end{bmatrix}\).

2. Suppose that

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 2 \\
0 & 1 & 0 & 0 \\
2 & 19 & 2 & 1
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
2 \\
4 \\
2 \\
1
\end{bmatrix}
\]

(a) What is the determinant of \(A\)?

(b) If \(Ax = b\), use Cramer’s rule to find \(x_2\).

3. Suppose that

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
b = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(a) Compute \(B = A^TA\) and find \(B^{-1}\).

(b) Explain why \(Ax = b\) is inconsistent, and write down the normal equations of the system.

(c) Find the least squares solution to \(Ax = b\).

(d) What is the projection of \(b\) onto \(\text{Col} \ A\)?

4. (a) Show that \(\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}\) and \(\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}\) are both bases for \(\mathbb{R}^2\).
(b) Write down the change of basis matrices \( P_{C \to B} \) and \( P_{B \to C} \).

(c) If \( v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), find both \([v]_C\) and \([v]_B\).

(d) Suppose that \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear transformation so that \( T(c_1) = b_1 + 2b_2 \) and \( T(c_2) = b_2 \). Find \([T]_C\), the matrix for \( T \) relative to \( C \).

5. (a) Are the three polynomials \( 1 + 2t, t + t^2, 3t^2 + 2t - 4 \) a basis for \( \mathbb{P}_2 \)? Explain why or why not.

(b) Every year 20% of the people in City A move to City B, and 10% of the people in City B move to city A. Suppose that initially, each city has 1,000,000 people.
How many people will live in each city after two years? After a very large number of years?

6. (a) Diagonalize the matrix \( A = \begin{bmatrix} 4 & -6 \\ 0 & 1 \end{bmatrix} \).

(b) Explain how to quickly compute \( A^{20} \). (You don’t need to actually do it.)

(c) Give the solution to the differential equation \( x' = Ax \) with \( x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \).

7. The matrix
\[
A = \begin{bmatrix}
4 & 20 & 8 & 16 & 16 & -3 & 23 \\
8 & 40 & 5 & 21 & 10 & -2 & 36 \\
5 & 25 & 2 & 12 & 4 & -1 & 21 \\
1 & 5 & 2 & 4 & 4 & -1 & 5
\end{bmatrix}
\]
can be row reduced to
\[
\begin{bmatrix}
1 & 5 & 0 & 2 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Give bases for \( \text{Row } A \), \( \text{Col } A \), and \( \text{Nul } A \).

(b) What is the rank of \( A \), and what are the dimensions of \( \text{Row } A^T \), \( \text{Col } A^T \), and \( \text{Nul } A^T \)?

8. (a) Find an \( LU \)-decomposition of the matrix \( X = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \).

(b) Use your answer to (a) to find the determinant of \( X \).

9. (a) Find a \( QR \)-decomposition of the matrix \( Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \).
(b) What is the orthogonal projection of \[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\] onto \(\text{Col} \, Y\)?

10. Consider the matrix
\[
H = \begin{bmatrix}
2 & 0 & 0 \\
3 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

(a) What are the eigenvalues of \(H\)? Find all of the eigenvectors for each eigenvalue.

(b) Is the matrix \(H\) diagonalizable? Explain.