1. Let \( \mathbf{v} \) be the vector \( \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Let \( W \) be the subspace of \( \mathbb{R}^3 \) given by all vectors perpendicular to \( \mathbf{v} \).

(a) Find a basis for \( W \) (hint: \( W \) is the nullspace of the \( 1 \times 3 \) matrix \( \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \)).

To find the nullspace of \( \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \), just use parametric vector form as usual. Reduced echelon form of the augmented matrix (with an extra column of 0s) is given by

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The variables \( x_2 \) and \( x_3 \) are free, while \( x_1 \) is a pivot. The general solution is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
-s-t \\
s \\
t
\end{bmatrix} = s
\begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix} + t
\begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}.
\]

A basis for \( W \) is given by the two vectors

\[
\mathbf{x}_1 = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}.
\]

(b) Find an orthonormal basis for \( W \).

The two vectors we got are not an orthonormal basis: they aren’t even orthogonal to each other. So we need to use the Gram–Schmidt process. First we construct an orthogonal basis:

\[
\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}
\]

\[
\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
-1/2 \\
-1/2 \\
1
\end{bmatrix}.
\]

This has \( \mathbf{v}_2 \cdot \mathbf{v}_1 = 0 \), like we want. But it’s still not an orthogonal basis: for that, we need to divide each of the vectors by its length.

\[
\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix}
-1/\sqrt{6} \\
-1/\sqrt{6} \\
\sqrt{2}/3
\end{bmatrix}.
\]
(c) The vector \[
\begin{bmatrix}
2 \\
-1 \\
-1
\end{bmatrix}
\] is perpendicular to \(v\), so it's in the subspace \(W\). Write this vector as a linear combinations of your orthonormal basis vectors from (b).

There's a quick rule for this: \(v = c_1u_1 + c_2u_2\) where

\[
c_1 = v \cdot u_1 = -2/\sqrt{2} - 1/\sqrt{2} = -\frac{3\sqrt{2}}{2}
\]

\[
c_2 = v \cdot u_2 = -2/\sqrt{6} + 1/\sqrt{6} - \sqrt{2}/3 = -\sqrt{3}/2.
\]

So

\[
v = -\frac{3\sqrt{2}}{2}u_1 - \frac{\sqrt{3}}{2}u_2.
\]

(d) Let \(p = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\). What point on \(W\) is closest to \(p\)?

We want \(\text{proj}_W p\), which is given by

\[
\text{proj}_W p = (p \cdot u_1)u_1 + (p \cdot u_2)u_2 = \cdots = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.
\]

(e) Let \(A\) be the \(3 \times 2\) matrix whose columns are the vectors in your answer to (a). Find a QR decomposition for \(A\).

We have \(Q\) given by the columns \(u_1\) and \(u_2\):

\[
Q = \begin{bmatrix}
-1/\sqrt{2} & -1/\sqrt{6} \\
1/\sqrt{2} & -1/\sqrt{6} \\
0 & \sqrt{2}/3
\end{bmatrix}.
\]

Then

\[
R = Q^TA = \begin{bmatrix}
-1/\sqrt{2} & 1/\sqrt{2} & 0 \\
-1/\sqrt{6} & 1/\sqrt{6} & \sqrt{2}/3 \\
-1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/3
\end{bmatrix} \begin{bmatrix}
-1 & -1 \\
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
\sqrt{2} & 1/\sqrt{2} \\
1 & 0 \\
0 & \sqrt{3}/2
\end{bmatrix}.
\]