A NOTE ON VALUES OF THE DEDEKIND ZETA-FUNCTION AT ODD POSITIVE INTEGERS

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Abstract. For an algebraic number field $K$, let $\zeta_K(s)$ be the associated Dedekind zeta-function. It is conjectured that $\zeta_K(m)$ is transcendental for any positive integer $m > 1$. The only known case of this conjecture was proved independently by C. L. Siegel and H. Klingen, namely that, when $K$ is a totally real number field, $\zeta_K(2n)$ is an algebraic multiple of $\pi^{2n[K:Q]}$ and hence, is transcendental. If $K$ is not totally real, the question of whether $\zeta_K(m)$ is irrational or not remains open. In this paper, we prove that for a fixed integer $n \geq 1$, at most one of $\zeta_K(2n+1)$ is rational, as $K$ varies over all imaginary quadratic fields. We also discuss a generalization of this theorem to CM-extensions of number fields.

1. Introduction

The Riemann zeta-function, $\zeta(s)$ has occupied center stage in mathematics since its introduction in the phenomenal 1859 paper of Riemann. In this paper, Riemann proved that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\Re(s) > 1$ has an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$ with residue 1, and satisfies a functional equation relating the value at $s$ to the value at $1 - s$. Moreover, it was proved independently by Hadamard and de la Vallée Poussin that the distribution of primes (in particular, the prime number theorem) is a consequence of the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$ together with the simple pole at $s = 1$. These ideas gave birth to the study of zeta and $L$-functions in other number theoretic contexts.

The focus of the current paper is another question about $\zeta(s)$, which has been baffling mathematicians since the 18th century. In 1735, Euler proved that for $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

where $B_k$ is the $k$-th Bernoulli number given by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}.$$

We recognize that the infinite series evaluated by Euler are nothing but the special values, $\zeta(2k)$. Whether such an “explicit evaluation” exists for the values $\zeta(2k + 1)$ as well, is still an open
question. Using the functional equation of $\zeta(s)$, Euler’s theorem implies that
\[
\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1} \quad \text{for } k \in \{0, 1, 2, \ldots\}
\]
Thus, the value of $\zeta(s)$ at negative integers is rational with $\zeta(-2n) = 0$ for $n \in \mathbb{N}$.

Furthermore, the transcendence of $\pi$ due to Lindemann implies that $\zeta(2k) \in \pi^{2k} \mathbb{Q}^\times$, is also transcendental for every $k \in \mathbb{N}$. On the other hand, the algebraic/transcendental nature of $\zeta(2k+1)$ is shrouded in mystery. Spectacular breakthroughs have recently been made by Apéry [1] in 1978 who showed that $\zeta(3) \notin \mathbb{Q}$; by Rivoal [16] in 2000 and Ball and Rivoal [3] in 2001, who showed that for infinitely many $k$, $\zeta(2k+1) \notin \mathbb{Q}$; and W. Zudilin [24] who proved that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

Let $K$ be a number field with $[K : \mathbb{Q}] = n$ and $\mathcal{O}_K$ be its ring of integers. Then the Dedekind zeta-function attached to $K$ is defined as
\[
\zeta_K(s) := \prod_{\mathfrak{p} \subseteq \mathcal{O}_K, \mathfrak{p} \neq 0} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1} = \sum_{\mathfrak{A} \subseteq \mathcal{O}_K, \mathfrak{A} \neq 0} \frac{1}{N\mathfrak{A}^s}, \quad \Re(s) > 1,
\]
where the product is over non-zero prime ideals in $\mathcal{O}_K$ and $N$ denotes the absolute norm. When $K = \mathbb{Q}$, the Dedekind zeta-function $\zeta_\mathbb{Q}(s)$ is simply the Riemann zeta-function $\zeta(s)$.

The function $\zeta_K(s)$ was introduced by R. Dedekind, who also conjectured its analytic continuation, which was proved later by Hecke [8]. The function $\zeta_K(s)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$. The residue at $s = 1$ is given by the analytic class number formula,
\[
\lim_{s \to 1^+} (s - 1) \zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} h R}{\omega \sqrt{|d_K|}},
\]
where $r_1$ is the number of real embeddings of $K$, $2r_2$ is the number of complex embeddings of $K$, $h$ denotes the class number, $R$ is the regulator, $\omega$ is the number of roots of unity in $K$ and $d_K$ is the discriminant of $K$ (see [14, Chapter 1]).

Analogous to the Riemann zeta-function, the Dedekind zeta-function captures crucial information about the distribution of prime ideals in $\mathcal{O}_K$. For example, the non-vanishing of $\zeta_K(s)$ on the line $\Re(s) = 1$ along with its simple pole at $s = 1$, implies the prime ideal theorem. The prime ideal theorem asserts that if $\pi_K(x) := \#\{\mathfrak{p} \in \mathcal{O}_K : \mathfrak{p} \text{ is prime}, N\mathfrak{p} \leq x\}$, then
\[
\pi_K(x) \sim \frac{x}{\log x},
\]
as $x \to \infty$. For a proof of this theorem, we refer the reader to the exposition in [14, Theorem 3.2].

The Dedekind zeta function satisfies a functional equation in the same spirit as the Riemann zeta-function, namely,
\[
\xi_K(s) = \xi_K(1 - s),
\]
where
\[
\xi_K(s) := \left(\frac{\sqrt{|d_K|}}{2^{r_2} \pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s),
\]
which is analytic in the entire complex plane except for simple poles at $s = 0$ and $s = 1$. Since the gamma function has poles at negative integers, we see that if $r_1 > 0$, then from the functional
equation one can deduce that \( \zeta_K(s) \) is always zero at all non-zero negative even integers. Additionally, if \( K \) is not totally real (i.e., \( r_2 > 0 \)), then \( \zeta_K(s) \) is zero at all odd negative integers as well. Thus, the only non-zero values of \( \zeta_K(s) \), at negative integers \( -m \), arise when \( K \) is totally real and \( m > 0 \) is odd. From the functional equation, these values correspond to \( \zeta_K(2^n) \) for an integer \( n > 0 \).

In 1940, Hecke [9] proved that \( \zeta_K(2n) \) is an algebraic multiple of \( \pi^{4n} \) for a real quadratic field \( K \). This led him to conjecture similar phenomena when \( K \) is any totally real field. Indeed, it was shown by C. L. Siegel and H. Klingen [11] independently, that when \( F \) is totally real, \( \zeta_F(1-2n) \) is rational. This translates to \( \zeta_F(2n) \) being an algebraic multiple of \( \pi^{2n[F:Q]} \), generalizing Euler’s 1737 theorem for the Riemann zeta-function. The method utilized by them relied on the theory of Hilbert modular forms. An accessible exposition of the proof can be found in the appendix of Siegel’s TIFR lecture notes [19]. In 1976, T. Shintani [17] provided an alternate proof of this theorem from a classical perspective, whereas geometric proofs have recently appeared in [5] and in [4].

When \( K \) is not totally real, nothing is known regarding the irrationality or transcendence of \( \zeta_K(n) \). In 1990, D. Zagier [23] put forth a conjecture connecting these values to the polylogarithm function,

\[
\text{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad |z| < 1.
\]

He conjectured that “\( \zeta_K(n) \) is a simple multiple of the determinant of a matrix whose entries are linear combinations of polylogarithms evaluated at a certain number in \( K \).” The case \( n = 3 \) of Zagier’s conjecture was settled by A. Goncharov [7]. However, we are still far from understanding the nature of these numbers.

If \( K/Q \) is an imaginary quadratic field, then \( \zeta_K(s) = \zeta(s) L(s, \chi) \), where \( \chi \) is an odd Dirichlet character. We know that \( L(2m+1, \chi) \) is an algebraic multiple of \( \pi^{2m+1} \) (see [15, Proposition 2]). Thus, \( \zeta_K(2m+1) = \zeta(2m+1) L(2m+1, \chi) \) is an algebraic multiple of \( \pi^{2m+1} \zeta(2m+1) \). We would anticipate all of these numbers to be transcendental, however, we are far from establishing this.

In particular, we expect all of the numbers \( \zeta_K(2m+1) \) (when \( K \) ranges over imaginary quadratic fields) to be irrational. We can prove:

**Theorem 1.1.** Let \( m \geq 1 \) be a fixed integer. Then the numbers

\[
\left\{ \zeta_K(2m+1) : K/Q \text{ is an imaginary quadratic extension} \right\}
\]

are irrational with at most one exception.

This immediately implies that

**Corollary 1.** For a positive real number \( D \), let

\[
\mathfrak{F}_D := \{ K : K/Q \text{ is an imaginary quadratic extension}, |d_K| \leq D \},
\]

where \( |d_K| \) denotes the absolute discriminant of the field \( K \). Then, Theorem 1.1 implies that

\[
\frac{\#\{ \zeta_K(2m+1) : \zeta_K(2m+1) \in \mathbb{Q}, K \in \mathfrak{F}_D \text{ and } m \in \mathbb{N}, m \leq x \}}{\#\{(m,K) : K \in \mathfrak{F}_D, m \in \mathbb{N}, m \leq x \}} \leq \frac{1}{|\mathfrak{F}_D|},
\]

and the right hand side tends to zero as \( D \to \infty \).
Additionally, as a consequence of Proposition 2.2 used in the proof of Theorem 1.1, we obtain the following interesting corollaries, which will be proved in Section 3.

**Corollary 2.** Let \( m \) be a fixed positive integer. Then, either all the numbers
\[
\left\{ \zeta_K(2m+1) : K/Q \text{ is an imaginary quadratic extension} \right\}
\]
are transcendental or all the above numbers are algebraic.

**Corollary 3.** Let \( m \) be a fixed positive integer. Then the numbers
\[
\left\{ \zeta_K(2m+1) : K/Q \text{ is an imaginary quadratic extension} \right\}
\]
are \( \mathbb{Q} \)-linearly independent.

A number field \( E \) is said to be CM if there exists a subfield \( F \) of \( E \), such that \( F \) is totally real, and \( E \) is a totally imaginary quadratic extension of \( F \). The aim of this paper is to highlight that an irrationality result for the values of the Dedekind zeta-function of CM-number fields can be deduced from our current knowledge. In particular, we obtain the following.

**Theorem 1.2.** Let \( m \geq 1 \) be a fixed integer. Fix a totally real field \( F \). Consider any family \( \mathfrak{F} \) of CM-extensions \( E/F \) satisfying the following: for \( E_1, E_2 \in \mathfrak{F} \) with \( E_1 \neq E_2 \), the square-free parts of \( d_{E_1} \) and \( d_{E_2} \) are co-prime. Then the numbers
\[
\left\{ \zeta_E(2m+1) : E \in \mathfrak{F} \right\}
\]
are irrational with at most one exception.

Our work will use the theory of Artin \( L \)-series and a central theorem of Coates and Lichtenbaum [6] regarding special values of certain Artin \( L \)-series.

2. Preliminaries

In this section, we review parts of algebraic number theory that are relevant to our discussion.

2.1. **Artin \( L \)-functions.** We summarize relevant facts regarding Artin \( L \)-functions below. A gentle introduction to Artin \( L \)-functions can be found in N. Snyder’s senior thesis, titled “Artin \( L \)-functions: A Historical Approach” [20]. A more concise account is included in the monograph [14, Chapter 2].

Let \( E/F \) be a Galois extension of number fields with Galois group \( G \). Let \( \rho : G \to GL(V) \) be a representation of \( G \) with character \( \chi \). Then the Artin \( L \)-function associated to the extension \( E/F \) and the representation \( \rho \) is defined as
\[
L(s, \chi, E/F) = \prod_{\substack{p \subseteq \mathcal{O}_F, \\ \text{p prime}}} L_p(s, \chi, E/F),
\]
where the local factors at each prime ideal \( p \) of \( \mathcal{O}_F \) are as follows. Suppose first that \( p \) is unramified in \( E \). Let \( \sigma_p \) denote the conjugacy class corresponding to the Frobenius at \( p \). The local factor at \( p \) is defined as
\[
L_p(s, \chi, E/F) = \det \left( I - \rho(\sigma_p)Np^{-s} \right)^{-1}.
\]
Now suppose that $p$ is ramified in $E$ and fix a prime $\mathfrak{P}$ above $p$. Let $V_{I\mathfrak{P}}$ be the subspace of vectors fixed by the inertia group $I_{\mathfrak{P}}$, pointwise. That is,
$$V_{I\mathfrak{P}} = \left\{ v \in V : \rho(\iota) \cdot v = v, \text{ for all } \iota \in I_{\mathfrak{P}} \right\}.$$ Since $I_{\mathfrak{P}}$ is a normal subgroup of $G_{\mathfrak{P}}$, one can see that $V_{I\mathfrak{P}}$ is $G_{\mathfrak{P}}$-invariant. Let $\sigma_{\mathfrak{P}}$ be any Frobenius element at $\mathfrak{P}$. Then,
$$L_p(s, \chi, E/F) = \det \left( I - \rho(\sigma_{\mathfrak{P}})|_{V_{I\mathfrak{P}}}, \mathcal{N}p^{-s} \right)^{-1},$$
where $\sigma|_{V_{I\mathfrak{P}}}$ denotes $\sigma$ restricted to the invariant subspace $V_{I\mathfrak{P}}$ for $\sigma \in G_{\mathfrak{P}}$. Note that the above definition is independent of the choice of the Frobenius element. The infinite product consisting of all these local factors converges absolutely for $\Re(s) > 1$ and defines the Artin $L$-function associated to $\rho$ and the extension $E/F$.

These $L$-functions take more familiar shape in certain scenarios. For example, suppose $E/F$ is Galois with Galois group $G$. Then, the Artin $L$-function obtained by considering the trivial representation of $G$ is nothing but the Dedekind zeta-function attached to the ground field, $\zeta_F(s)$. On the other hand, the Artin $L$-functions associated to characters of $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ are precisely the Dirichlet $L$-functions.

Artin conjectured that any Artin $L$-function $L(s, \chi, E/F)$ associated to a character $\chi$ of $\text{Gal}(\mathcal{F}/F)$ extends to an analytic function to the entire complex plane except for a possible pole at $s = 1$, of order equal to the multiplicity of the trivial representation in $\chi$. This is one of the classical conjectures in number theory and remains unresolved in general. It is known in the special case when $\text{Gal}(E/F)$ is abelian. In this case, by Artin’s reciprocity law, the Artin $L$-function of an irreducible character corresponds to a Hecke $L$-function, which is known to be entire (see [12, Chapter 9] for further details). There are also some recent results in the 2-dimensional case due to Langlands [13], Tunnell [21], Khare and Wintenberger [10].

Artin $L$-functions satisfy a functional equation in the same spirit as the Riemann zeta-function. At the infinite primes, i.e., the Archimedean places, the corresponding Euler factors are defined as follows. Let $\nu$ be an Archimedean place of $F$. Then,
$$L_\nu(s, \chi, E/F) = \begin{cases} ((2\pi)^{-s} \Gamma(s))^{\dim(\rho)}, & \text{if } \nu \text{ is complex,} \\ (\pi^{-s/2} \Gamma(s/2))^a (\pi^{-(s+1)/2} \Gamma((s + 1)/2))^b & \text{if } \nu \text{ is real.} \end{cases}$$
Here $a$ is the dimension of the $+1$ eigenspace of complex conjugation and $b$ is the dimension of $-1$ eigenspace of complex conjugation. Hence,
$$a + b = \dim(\rho).$$
Therefore, the gamma factors for $L(s, \rho, E/F)$ are
$$\gamma(s, \chi, E/F) = \prod_{\nu \text{ - Archimedean place of } F} L_\nu(s, \chi, E/F).$$
An important invariant that makes an appearance in the functional equation is the Artin conductor, $f_\chi$. The Artin conductor is an ideal in the ring $\mathcal{O}_F$ and is defined by the restriction of $\chi$ to the inertia group and its various subgroups. We refrain from giving the technical definition here and refer the reader to [14, pg. 28] for the precise version. However, we note one of the useful connections of the Artin conductor to the relative discriminants of number fields. In 1931,
E. Artin [2] proved the conductor-discriminant formula for any Galois extension of number fields $E/F$. This formula states that

$$\mathfrak{D}_{E/F} = \prod_{\chi \in \hat{G}} f^{(1)}_{\chi}, \quad (1)$$

where $\mathfrak{D}_{E/F}$ denotes the relative discriminant of $E/F$, $\hat{G}$ denotes the set of all irreducible characters of $G$ and $\chi(1)$ is the dimension of the irreducible representation corresponding to $\chi$.

Let

$$A_\chi = |d_F|^{\chi(1)} Nf_\chi \in \mathbb{Q},$$

where $d_F$ denotes the discriminant of the field $F$. The completed Artin $L$-function can then be defined as

$$\Lambda(s, \chi, E/F) := A_\chi^{s/2} \gamma(s, \chi, E/F) L(s, \chi, E/F).$$

This completed Artin $L$-function satisfies the functional equation

$$\Lambda(s, \chi, E/F) = W(\chi) \Lambda(1 - s, \chi, E/F), \quad (2)$$

for all $s \in \mathbb{C}$. The number $W(\chi)$ is called the Artin root number and is a complex number of absolute value 1, carrying deep arithmetic meaning. One important observation here is that if $\chi$ is real-valued, then $W(\chi) = \pm 1$. This can be seen by comparing the above functional equation with its complex conjugate.

Using basic functorial properties of Artin $L$-functions, one can translate the group theoretic identity,

$$\text{reg}_G = \sum_{\chi \in \hat{G}} \chi(1) \chi,$$

to a factorization identity, namely,

$$\zeta_E(s) = \zeta_F(s) \prod_{\chi \in \hat{G}, \chi \neq 1} L(s, \chi, E/F)^{\chi(1)}, \quad (3)$$

where $\text{reg}_G$ denotes the regular representation of $G$.

2.2. Values of zeta-functions at negative integers. Let $\zeta(s)$ denote the Riemann zeta-function. As a by-product of Riemann’s proof of analytic continuation and functional equation of $\zeta(s)$, one can obtain the evaluation of $\zeta(-n)$ for a positive integer $n$ in terms of Bernoulli numbers. Analogously, for a positive integer $q$, let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character mod $q$ and let $L(s, \chi)$ be the Dirichlet $L$-series attached to $\chi$. It can be shown that $L(s, \chi)$ is entire when $\chi$ is non-principal. Furthermore, for any integer $n \geq 0$,

$$L(-n, \chi) = -\frac{q^n}{n+1} \sum_{a=1}^{q} \chi(a) B_{n+1} \left( \frac{a}{q} \right),$$

where $B_n(X) \in \mathbb{Q}[X]$ is the $n^{th}$ Bernoulli polynomial. We refer the reader to [22, Chapter 4] for a proof. This implies that if $\chi$ is a quadratic character, then $L(-n, \chi) \in \mathbb{Q}$.

Similarly, the Siegel-Klingen theorem proves that the values of Dedekind zeta-functions attached to totally real fields at odd negative integers are rational. Moreover, Siegel [18] proved an analogue of this theorem for Hecke $L$-series associated to ray class characters. It was further suggested by Serre that Siegel’s work itself implies a similar result for all Artin $L$-functions. This
appears in the paper of Coates and Lichtenbaum [6, Theorem 1.2]. In particular, they show the following.

**Theorem 2.1** (Coates-Lichtenbaum). Let $F$ be a totally real number field and $E/F$ be a Galois extension with Galois group $G$. Let $\rho$ be a representation of $G$ with character $\chi$ and $L(s,\chi,E/F)$ be the associated Artin $L$-function. Let $Q(\chi) = Q(\{\chi(g) : g \in G\})$ and $n$ be a positive integer such that $L(-n,\chi,E/F) \neq 0$. Then $L(-n,\chi,E/F)$ is an algebraic number lying in the field $Q(\chi)$.

It is evident from the functional equation (2) that there exist positive integers $n$ such that $L(-n,\chi,E/F)$ is not zero if and only if $F$ is totally real and either (a) $E$ is totally real and $n$ is odd or (b) $E$ is totally imaginary and $n$ is even.

**2.3. An important proposition.** The proof of our theorem is based on the following proposition. In order to state the proposition, we define the notion of rational equivalence. Two complex numbers $\alpha$ and $\beta$ are said to be rationally equivalent, i.e., $\alpha \sim_Q \beta$ if $\beta = u\alpha$ for some $u \in Q^\times$. With this definition, we show that

**Proposition 2.2.** Fix a totally real number field $F$. Let $E_1$ and $E_2$ be two CM-extensions of $F$ and $d_{E_1}$ and $d_{E_2}$ be their respective discriminants. Then, for any fixed integer $m > 0$,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_Q \left( \frac{|d_{E_2}|}{|d_{E_1}|} \right)^{1/2}.$$

**Proof.** Let $G_j := \text{Gal}(E_j/F)$ for $j = 1, 2$. Then we have, $G_j = \{1, c_j\}$, where $c_j$ denote complex conjugation. Let the characters corresponding to $c_j$ be $\chi_j : G_j \to \{\pm 1\}$ where $\chi_j(c_j) = -1$. By the factorization (3),

$$\zeta_{E_j}(s) = \zeta_F(s)L(s,\chi_j,E_j/F), \quad j = 1, 2.$$

Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} = \frac{L(2m+1,\chi_1,E_1/F)}{L(2m+1,\chi_2,E_2/F)}.$$

The functional equation of Artin $L$-functions (2) relate the value at $2m+1$ with the value at $-2m$. Since $F$ is totally real, all archimedean places of $F$ are real and hence, none of the gamma factors appearing in the functional equation have poles at these integers. Thus, for $j = 1, 2$, we have

$$A_{\chi_j}^{(2m+1)/2} \gamma(2m+1,\chi_j,E_j/F) L(2m+1,\chi_j,E_j/F)$$

$$= W(\chi_j) A_{\chi_j}^{-m} \gamma(-2m,\chi_j,E_j/F) L(-2m,\chi_j,E_j/F).$$

This implies that

$$L(2m+1,\chi_j,E_j/F) = W(\chi_j) (A_{\chi_j})^{-2m-1/2} \gamma(-2m,\chi_j,E_j/F) \gamma(2m+1,\chi_j,E_j/F) L(-2m,\chi_j,E_j/F).$$

Since $E_j$ is a totally imaginary extension of $F$, Theorem 2.1 and (2) implies that

$$0 \neq L(-2m,\chi_j,E_j/F) \in Q(\chi_j) = Q.$$

We would like to remark here that we do not need the full generality of Theorem 2.1. Since an $L$-function associated to a one dimensional character is a product of Hecke $L$-functions, Siegel’s result would suffice for the above conclusion.

Moreover, $W(\chi_j) = \pm 1$ because $\chi_j$ are real valued for $j = 1, 2$. Hence,

$$L(2m+1,\chi_j,E_j/F) \sim_Q A_{\chi_j}^{-1/2} \gamma(-2m,\chi_j,E_j/F) \gamma(2m+1,\chi_j,E_j/F).$$

(4)
On taking the ratio of $L(2m+1, \chi_j, E_j/F)$ for $j = 1, 2$, the gamma factors cancel as they are same for both the Artin $L$-functions under consideration. Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_{Q} \left( \frac{A_{X_2}}{A_{X_1}} \right)^{1/2}.$$ 

The factor of $|d_F|$ will be common to both the values, and disappears in the ratio. Thus, the contributing factor reduces to

$$\frac{N_F/Q \chi_1}{N_F/Q \chi_2}.$$ 

Since the conductor of the trivial representation is the unit ideal, the conductor-discriminant formula (1) implies

$$f_{\chi_j} = D_{E_j/F}.$$ 

The relative discriminant of $E_j/F$ is related to the absolute discriminant of $E_j$ by the formula

$$d_{E_j} = d_F^2 \cdot N_{F/Q} D_{E_j/F}.$$ 

The statement of the proposition now follows. □

3. **Proof of the Main theorems**

3.1. **Proof of Theorem 1.1.** By Proposition 2.2, we know that if $K_1$ and $K_2$ are two imaginary quadratic extensions of $\mathbb{Q}$, then

$$\frac{\zeta_{K_1}(2m+1)}{\zeta_{K_2}(2m+1)} \sim_{Q} \left( \frac{|d_{K_2}|}{|d_{K_1}|} \right)^{1/2}.$$ 

Since $K_1$ and $K_2$ are distinct quadratic extensions of $\mathbb{Q}$, and $K_j = \mathbb{Q} (\sqrt{|d_{K_j}|})$, the numbers $\sqrt{|d_{K_j}|}$ are not rational multiples of each other. Hence, the above quotient is irrational, proving the theorem.

3.2. **Proof of Corollary 2.** The statement of Proposition 2.2 implies that for two distinct imaginary quadratic fields, $K_1$ and $K_2$, the quotient

$$\frac{\zeta_{K_1}(2m+1)}{\zeta_{K_2}(2m+1)} \in \overline{\mathbb{Q}}.$$ 

Therefore, if $\zeta_K(2m+1) \in \overline{\mathbb{Q}}$ for some imaginary quadratic field $K$, then the same will be true for all imaginary quadratic fields.

3.3. **Proof of Corollary 3.** Suppose that there exist imaginary quadratic fields $K_j/\mathbb{Q}$, $1 \leq j \leq r$ such that

$$\sum_{j=1}^{r} c_j \zeta_{K_j}(2m+1) = 0, \quad c_j \in \mathbb{Q}, \quad 1 \leq j \leq r.$$ 

By the factorization (3), we have $\zeta_{K_j}(2m+1) = \zeta(2m+1) L(2m+1, \chi_j)$. Hence, the above relation reduces to a relation among the values of Dirichlet $L$-functions. Now take $F = \mathbb{Q}$, $E_j = K_j$ in (4). Note that the corresponding ratio of gamma factors is independent of $\chi_j$ and simplifies to

$$\frac{\gamma(-2m, \chi_j, K_j/\mathbb{Q})}{\gamma(2m+1, \chi_j, K_j/\mathbb{Q})} \sim_{Q} \left( \frac{\pi^{(m+\frac{1}{2})}}{\pi^{-(m+1)}} \right) \frac{\Gamma\left(-m+\frac{1}{2}\right)}{\Gamma(m+1)} \sim_{Q} \frac{\pi^{(2m+\frac{1}{2})}}{\pi^{2m+1}},$$
as $\Gamma(1/2) = \sqrt{\pi}$. Thus, the above relation becomes

$$\sum_{j=1}^{r} \frac{R_j}{\sqrt{|d_{K_j}|}} = 0,$$

for certain rational numbers $R_j$. However, the numbers $\sqrt{|d_{K_j}|}$ are $\mathbb{Q}$-linearly independent. This proves the corollary.

3.4. **Proof of Theorem 1.2.** The conditions on the family $\mathfrak{F}$ ensure that for any $E_1$ and $E_2$ in $\mathfrak{F}$, $|d_{E_1}|/|d_{E_2}|$ is not a perfect square in $\mathbb{Q}$. The corollary is now immediate from Proposition 2.2.

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