A COLD-BLOODED DEPENDENT TYPE THEORY

TESLA ZHANG

July 24, 2021

ABSTRACT. This document is a by-product of a cold-blooded introduction to categorical semantics of type theory.

1. JUDGMENTS

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A \text{ type}}{\Gamma, A \vdash} \qquad\qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A \text{ type}}{\Gamma \vdash u : A}$$

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A \text{ type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : A}{\Gamma \vdash u = v : A} \qquad\qquad \frac{\Gamma \vdash \qquad \Delta \vdash}{\Gamma \vdash \sigma : \Delta}$$

2. TYPE THEORY

$$\frac{\Gamma \vdash \qquad \Delta \vdash \qquad \Delta \vdash A \text{ type} \qquad \Gamma \vdash \sigma : \Delta \qquad \Gamma \vdash a : A\sigma}{\Gamma \vdash (\sigma, a) : (\Delta, A)}$$

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A \text{ type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash B \text{ type}}{\Gamma, B \vdash u : A}$$

$$\frac{\dfrac{\Gamma \vdash \qquad \Delta \vdash}{\Gamma \vdash \sigma : \Delta} \qquad \dfrac{\Delta \vdash A \text{ type}}{\Delta \vdash u : A}}{\Gamma \vdash A\sigma \text{ type} \qquad \text{and} \qquad \Gamma \vdash u\sigma : A\sigma} \qquad\qquad \frac{\dfrac{\Gamma \vdash \qquad \Gamma' \vdash}{\Gamma \vdash \sigma : \Gamma'} \qquad \dfrac{\Gamma' \vdash \qquad \Gamma'' \vdash}{\Gamma' \vdash \gamma : \Gamma''}}{\Gamma \vdash \gamma\sigma : \Gamma''}$$

$$\frac{}{\vdash \top \text{ type}} \qquad \frac{}{\vdash \star : \top} \qquad \frac{\vdash u : \top}{\vdash u = \star : \top} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type}}{\Gamma \vdash (A \times B) \text{ type}}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash (A \times B) \text{ type} \qquad \Gamma \vdash u : A \times B}{\Gamma \vdash u.1 : A \qquad \text{and} \qquad \Gamma \vdash u.2 : B}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : B}{\Gamma \vdash \langle u, v \rangle : A \times B}$$

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash B \text{ type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : B}{\Gamma \vdash \langle u, v \rangle.1 = u : A \qquad \text{and} \qquad \Gamma \vdash \langle u, v \rangle.2 = v : B}$$

$$\frac{\Gamma \vdash (A \times B)\ \text{type} \qquad \Gamma \vdash t : A \times B}{\Gamma \vdash t = \langle t.1, t.2 \rangle : A \times B} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type}}{\Gamma \vdash (A \sqcup B)\ \text{type}}$$

$$\frac{\Gamma \vdash (A \sqcup B)\ \text{type} \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{inl}(u) : A \sqcup B}$$

$$\frac{\Gamma \vdash (A \sqcup B)\ \text{type} \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma \vdash u : B}{\Gamma \vdash \mathsf{inr}(u) : A \sqcup B}$$

$$\frac{\begin{array}{c} \Gamma \vdash (A \sqcup B)\ \text{type} \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type} \\ \Gamma \vdash C\ \text{type} \qquad \Gamma, A \vdash u : C \qquad \Gamma, B \vdash v : C \qquad \Gamma \vdash t : A \sqcup B \end{array}}{\Gamma \vdash \mathsf{match}(t, u, v) : C}$$

$$\frac{\begin{array}{c} \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma \vdash C\ \text{type} \\ \Gamma, A \vdash u : C \qquad \Gamma, B \vdash v : C \qquad \Gamma \vdash t : A \end{array}}{\Gamma \vdash \mathsf{match}(\mathsf{inl}(t), u, v) = ut : C}$$

$$\frac{\begin{array}{c} \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma \vdash C\ \text{type} \\ \Gamma, A \vdash u : C \qquad \Gamma, B \vdash v : C \qquad \Gamma \vdash t : B \end{array}}{\Gamma \vdash \mathsf{match}(\mathsf{inr}(t), u, v) = vt : C} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type}}{\Gamma \vdash (A \to B)\ \text{type}}$$

$$\frac{\Gamma \vdash (A \to B)\ \text{type} \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \lambda.u : A \to B}$$

$$\frac{\begin{array}{c} \Gamma \vdash (A \to B)\ \text{type} \\ \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma \vdash u : A \to B \qquad \Gamma \vdash v : A \end{array}}{\Gamma \vdash \mathsf{ap}(u, v) : B}$$

$$\frac{\Gamma, A \vdash \qquad \Gamma \vdash B\ \text{type} \qquad \Gamma, A \vdash u : B \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{ap}(\lambda.u, v) = uv : B}$$

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash (\mathrm{Id}_A\ a\ b)\ \text{type}} \qquad \frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}_a : \mathrm{Id}_A\ a\ a}$$

$$\frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : \mathrm{Id}_A\ a\ b}{\Gamma \vdash a = b : A}$$

$$\frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash p : \mathrm{Id}_A\ a\ a}{\Gamma \vdash p = \mathsf{refl}_a : \mathrm{Id}_A\ a\ a}$$

$$\frac{\begin{array}{c} \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A \\ \Gamma \vdash p : \mathrm{Id}_A\ a\ b \qquad \Gamma \vdash q : \mathrm{Id}_A\ a\ b \end{array}}{\Gamma \vdash p = q : \mathrm{Id}_A\ a\ b} \qquad \frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma, A \vdash B\ \text{type}}{\Gamma \vdash (\Pi_A B)\ \text{type}}$$

$$\frac{\Gamma \vdash (\Pi_A B)\ \text{type} \qquad \Gamma, A \vdash B\ \text{type} \qquad \Gamma, A \vdash u : B}{\Gamma \vdash \Lambda.u : \Pi_A B}$$

$$\frac{\Gamma \vdash (\Pi_A B)\ \text{type} \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma, A \vdash B\ \text{type} \qquad \Gamma \vdash u : \Pi_A B \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{ap}(u, v) : Bv}$$

$$\frac{\Gamma \vdash (\Pi_A B)\ \text{type} \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma, A \vdash B\ \text{type} \qquad \Gamma, A \vdash u : B \qquad \Gamma \vdash v : A}{\Gamma \vdash \mathsf{ap}(\Lambda.u, v) = uv : Bv}$$

$$\frac{}{\vdash \mathrm{Prop}\ \text{type}} \qquad \frac{\Gamma \vdash p : \mathrm{Prop}}{\Gamma \vdash \mathrm{El}(p)\ \text{type}} \qquad \frac{\Gamma \vdash p : \mathrm{Prop} \qquad \Gamma \vdash u : \mathrm{El}(p) \qquad \Gamma \vdash v : \mathrm{El}(p)}{\Gamma \vdash u = v : \mathrm{El}(p)}$$

$$\frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash p : \Pi_{x:A}\Pi_{y:A}\mathrm{Id}_A\ x\ y}{\Gamma \vdash R(A, p) : \mathrm{Prop}}$$

$$\frac{\begin{array}{cc}\Gamma \vdash A\ \text{type} & \Gamma \vdash p : \Pi_{x:A}\Pi_{y:A}\mathrm{Id}_A\ x\ y \\ \Gamma \vdash B\ \text{type} & \Gamma \vdash q : \Pi_{x:B}\Pi_{y:B}\mathrm{Id}_B\ x\ y\end{array}}{\Gamma \vdash R(A, p) = R(B, q) : \mathrm{Prop}} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathrm{Bool}\ \text{type}}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{true} : \mathrm{Bool} \qquad \text{and} \qquad \Gamma \vdash \mathsf{false} : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : A \qquad \Gamma \vdash t : \mathrm{Bool}}{\Gamma \vdash \langle t\ ?\ u : v\rangle : A}$$

$$\frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash u : A \qquad \Gamma \vdash v : A \qquad \Gamma \vdash t : \mathrm{Bool}}{\Gamma \vdash \langle \mathsf{true}\ ?\ u : v\rangle = u : A \qquad \text{and} \qquad \Gamma \vdash \langle \mathsf{false}\ ?\ u : v\rangle = u : A}$$

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type}}{\Gamma \vdash ||A||\ \text{type}} \qquad \frac{\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash u : A}{\Gamma \vdash |u| : ||A||}$$

$$\frac{\begin{array}{c}\Gamma \vdash A\ \text{type} \qquad \Gamma \vdash ||A||\ \text{type} \\ \Gamma \vdash P : \mathrm{Prop} \qquad \Gamma \vdash f : A \to \mathrm{El}(P) \qquad \Gamma \vdash u : ||A||\end{array}}{\Gamma \vdash ||\_||\text{-}\mathsf{elim}(u, P, f) : \mathrm{El}(P)}$$

$$\frac{\Gamma \vdash \qquad \Gamma \vdash A\ \text{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathrm{Id}'_A\ a\ b : \mathrm{Prop}}$$

The Pennsylvania State University
Email address: yqz5714@psu.edu